Volatility Smiles and Yield Frowns

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Interest Rates and Volatility

- Practitioners and academics have both noticed similarities between interest rate modeling and volatility modeling.

  "There is a fundamental similarity between the role of interest rates in the pricing of bonds and the role of volatility in the pricing of index options." – Emanuel Derman et. al. (Investing in Volatility).

  "This note explores the analogy between the dynamics of the interest rate term structure and the implied volatility surface of a stock." – Rogers and Tehranchi.

Volatility Smiles and Yield Frowns

- A simple benchmark model for pricing zero coupon bonds can be used to define the concept of Yield to Maturity, which can be used in more complicated models.
- Analogously, a simple benchmark model for pricing European-style vanilla options can be used to define the concept of Implied Volatility by Moneyness, which can also be used in more complicated models.
- When implied volatilities are plotted against some moneyness measure, say strike minus spot, the resulting graph is typically convex, hence the phrase "Volatility Smile".
- Analogously, when bond yields are plotted against the bond's term, the resulting graph is typically concave. We call this result "Yield Frown".

Overview of Volatility Smiles and Yield Frowns

- We first review an overly simplistic benchmark model for pricing zero coupon bonds and a second overly simplistic model for pricing European options.
- The benchmark model for pricing bonds assumes that the short interest rate is constant, while the benchmark model for pricing options analogously assumes that the short variance rate of the underlying is constant.
- We then propose a new market model for pricing bonds and a second new market model for pricing options. In each market model, implied rates become stochastic.
- The two market models can be used to respectively determine an entire yield frown and an entire vol smile.
- In the bond market model, yield is quadratic in term, opening down. In the option market model, the implied variance rate is quadratic in moneyness, opening up. While both market models can still be improved upon, they provide a superior launching point than the benchmark models.
- We provide mathematical explanations for the similarities and differences between these results.

Benchmark Model for Pricing Zero Coupon Bonds

- We always work in continuous time with $t = 0$ as the valuation time.
- Let r_t be the continuously compounded short interest rate at time $t \geq 0$.
- In the benchmark model for pricing zero coupon bonds, the short interest rate is constant:

$$r_t = r, \qquad t \geq 0.$$

- We allow r to be any real number. At this time, short interest rates are positive in the United States and negative in Japan.

The Bond Pricing Formula and its Properties

- In the benchmark model of a constant short interest rate r, the zero coupon bond pricing formula is given by:

$$B^c(r,\tau) = e^{-r\tau}, \qquad r \in \mathbb{R}, \tau \geq 0.$$

- The superscript c in B^c is a reminder that the interest rate is assumed constant.
- The function B^c is positive and decreasing in r.
- The function B^c is strictly convex in r for $\tau > 0$, which will become important when we randomize r.

The Bond Price Process in the Benchmark Model

- Let $B_t(T)$ be the price at time $t \geq 0$ of a zero coupon bond, paying one dollar at its fixed maturity date $T \geq t$.
- In the benchmark model of a constant short interest rate r, the bond price process is given by:

$$B_t(T) = B^c(r, T - t) = e^{-r(T-t)}, \qquad t \in [0, T].$$

- Notice that if $r \neq 0$, then the bond price moves over time. This is called pull to par.
- Practitioners have developed a concept called yield to maturity which does not move in the benchmark model. We define this concept on the next slide.

Definition of Yield to Maturity

- Recall that in the benchmark model of a constant short interest rate r, the bond pricing formula is

$$B^c(r, \tau) = e^{-r\tau}, \qquad r \in \mathbb{R}, \tau \geq 0.$$

- Let $b_t(T) > 0$ be the time t market price of a bond paying one dollar at its fixed maturity date $T \geq t$.
- The bond's yield to maturity $y_t(T)$ is defined as the solution to:

$$b_t(T) = B^c(y_t(T), T - t) = e^{-y_t(T)(T-t)}, \qquad t \in [0, T].$$

- Inverting this expression for $y_t(T)$ give the following explicit formula for yield to maturity:

$$y_t(T) = -\ln b_t(T)/(T - t), \qquad t \in [0, T].$$

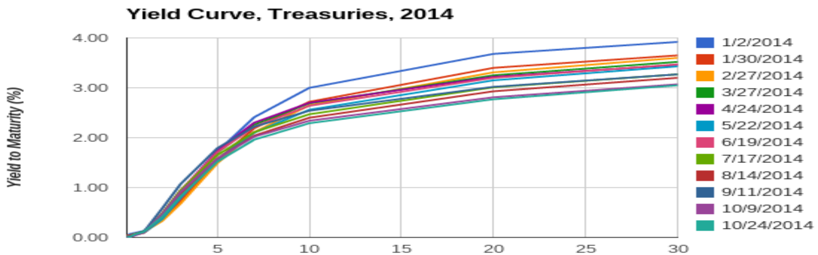
- In the benchmark model, the yield curve is both flat in T and static in t:

$$y_t(T) = r, \qquad t \in [0, T].$$

- In the benchmark model with $r \neq 0$, bond prices change over time while yields do not.

Average Shape of the Yield Curve

- On average, yields have been a concave function of term τ, defined as $T - t$.
- In fact, yields rose with term at a decreasing rate for each month in 2014:
- Clearly, we need a model that does not predict that the yield curve is flat.



Linking Yields to Pull to Par

- As its name suggests, yield to maturity (YTM) is the return from a "buy and hold" of a bond to its maturity. However, YTM has a 2nd financial meaning arising from a "buy then sell" strategy, which is key for us.
- The logarithmic derivative of the bond pricing formula $B^c(r, T-t) = e^{-r(T-t)}$ w.r.t. time t is:

$$\frac{\partial}{\partial t} \ln B^c(r, T-t) = \frac{\frac{\partial}{\partial t} B^c(r, T-t)}{B^c(r, T-t)} = r.$$

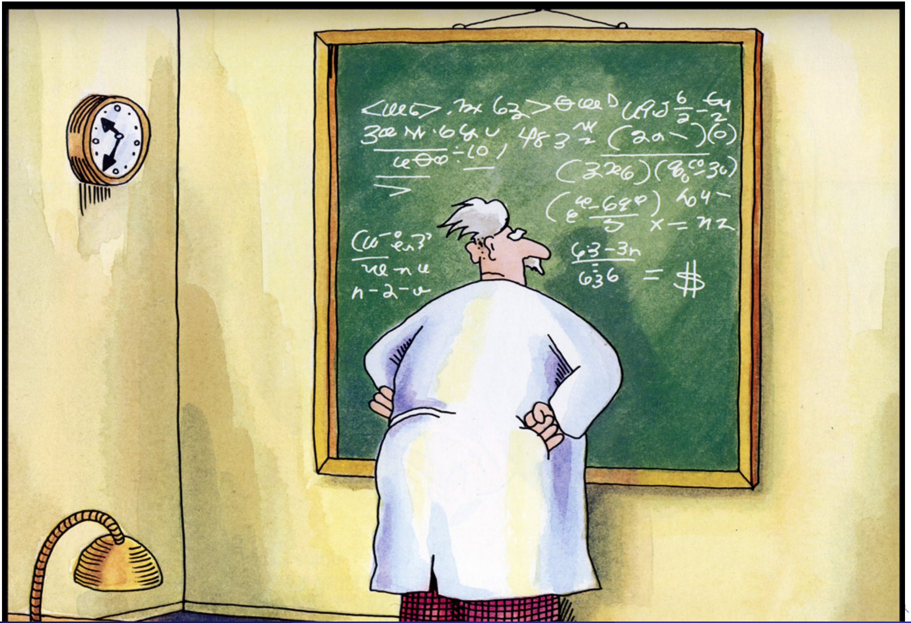
- If we now evaluate at $r = y_t(T)$:

$$\frac{\frac{\partial}{\partial t} B^c(r, T-t)\Big|_{r=y_t(T)}}{b_t(T)} = y_t(T), \qquad t \geq 0, T > t.$$

  Hence, YTM is also the theta of a 1\$ investment in bonds. YTM is the time component when attributing the P&L from investing \$1 in a bond and then selling immediately afterwards.

- So when interest rates are positive, time is money.

Einstein Discovers That Time Really is Money



Benchmark Option Pricing Model

- Three years before Einstein explained Brownian motion, Bachelier used this process to describe the price of an asset underlying an option. We will use Bachelier's option pricing model as a benchmark.
- We now assume zero interest rates. We also assume that the spot price S of the call's underlying asset has a positive short term variance rate which is constant through time at $a^2 > 0$. Thus $S_t = S_0 + aW_t, t \geq 0$, where W is a standard Brownian motion.
- Let $C^b(S-K, a, T-t)$ be the Bachelier model value of a European call paying $(S_T - K)^+$ at its maturity date T. Then Bachelier (1900) showed:

$$C^b(S-K, a, T-t) = a\sqrt{\tau}N'\left(\frac{x}{a\sqrt{\tau}}\right) + xN\left(\frac{x}{a\sqrt{\tau}}\right),$$

where $x = S - K$, $\tau = T - t$, and $N(z) \equiv \int_{-\infty}^{z} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$ is the standard normal distribution function.

Important Features of Bachelier's Call Pricing Formula

- Recall that with a^2 as the constant variance rate, $x = S - K$ as the excess of spot S over strike K, and $\tau = T - t$ as term, Bachelier's call value is:

$$C^b(x, a, \tau) = a\sqrt{\tau}N'\left(\frac{x}{a\sqrt{\tau}}\right) + xN\left(\frac{x}{a\sqrt{\tau}}\right), \qquad x \in \mathbb{R}, a > 0, \tau > 0,$$

where $N(z) \equiv \int_{-\infty}^{z} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$ is the standard normal distribution function.

- The function $C^b > 0$ is increasing in all 3 of its arguments. C^b is strictly convex in x for each $a > 0$ and $\tau > 0$, while C^b is strictly convex in a for each $x \neq 0$ and $\tau > 0$. The strict convexity of C^b in a will be important when we later randomize volatility.
- The second x derivative of C^b is called gamma:

$$\Gamma^b(x, a, \tau) \equiv C^b_{11}(x, a, \tau) = \frac{N'\left(\frac{x}{a\sqrt{\tau}}\right)}{a\sqrt{\tau}} > 0, \qquad x \in \mathbb{R}, a > 0, \tau > 0.$$

- A unit gamma option position will be analogous to a 1\$ investment in a bond.

Properties of the Call Price Process in Bachelier's Model

- Recall that in the benchmark bond pricing model with a nonzero interest rate, the forward movement of calendar time causes the price of a bond with a fixed maturity date to change.
- Analogously, in Bachelier's model, as the underlying's spot price moves, the price of a call at a fixed strike changes.
- As a result, practitioners have developed a concept analogous to yield called (normal) implied volatility, which is defined on the next slide.
- This concept is the current market standard for swaptions.

Definition of Normal Implied Volatility

- Recall again that with $a^2 > 0$ as the constant short variance rate, $x = S - K$ as the excess of the spot price S over the strike price K, and $\tau = T - t$ as the time to maturity, the Bachelier call value function is:

$$C^b(x, a, \tau) = a\sqrt{\tau}N'\left(\frac{x}{a\sqrt{\tau}}\right) + xN\left(\frac{x}{a\sqrt{\tau}}\right), \qquad x \in \mathbb{R}, a > 0, \tau > 0,$$

  where $N(z) \equiv \int\limits_{-\infty}^{z} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$ is the standard normal distribution function.

- When the market price of a call of a fixed maturity date $T > 0$ is known at time $t \in [0, T)$ to be $c_t(K) > (S - K)^+$, then the normal implied volatility $\eta_t(K)$ is defined as the positive solution to the equation:

$$c_t(K) = C^b\left(S_t - K, \eta_t(K), T - t\right), \qquad K \in \mathbb{R}, t \in [0, T].$$

  Since the function $C(x, a, \tau)$ is increasing in a, the inverse map relating $\eta_t(K)$ to $c_t(K)$ for each K exists, but is not explicit.

Normal Implied Volatility vs. Strike in Bachelier's Model

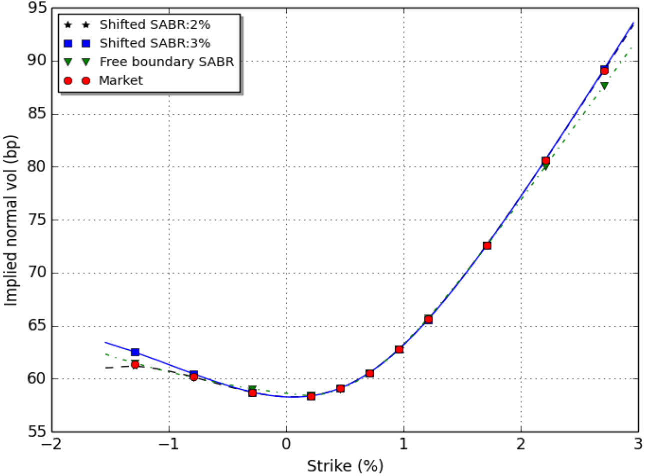
- We consider the relationship between a call's normal implied vol $\eta_t(K)$ and its strike price $K \in \mathbb{R}$ for a fixed maturity date $T > t$.
- In the benchmark option pricing model, the IV curve is both flat and static:

$$\eta_t(K) = a, \qquad t \in [0, T].$$

- In the benchmark model, call prices change over time while implied volatilities do not.
- It has become standard practice in swaptions markets to work with normal implied volatilities, rather than call swaption prices, even though both change over time in practice.

Average Shape of the Swaption Implied Vol Curve

- Normal implied vol's of swaptions are typically convex in the strike rate K.



Interpreting Halved Implied Variance Rates

- Clearly, we need a model that does not predict that the IV curve is flat.
- Recall that $y_t(T) = \frac{\frac{\partial}{\partial t} B^c(r, T-t)|_{r=y_t(T)}}{b_t(T)}$, $t \geq 0$, $T > t$. The yield at time t is just the time component when attributing the P&L from investing \$1 in a bond at time t and then selling immediately afterwards.
- The analogous equation for the halved (normal) implied variance rate at time t and strike K is:

$$\frac{1}{2}\eta_t^2(K) = -\frac{\frac{\partial}{\partial t} C_t^b(S-K, a, T-t)|_{S=S_t, a=\eta_t(K))}}{\Gamma_t(K)}, \qquad t \geq 0, K \in \mathbb{R}, T > t,$$

  where $C^b(S-K, a, T-t)$ is Bachelier's call pricing formula, and $\Gamma_t(K)$ is its 2nd derivative in S. The halved implied variance rate at time t negates the time component when attributing the P&L from a unit gamma investment in options at time t followed by an immediate sale.
- Under positive interest rates, time is money for a bondholder. Under positive variance rates, time is the enemy of an options holder. YTM & halved implied variance rate measure the size of the gains & losses respectively.

Concave Yield Curves and Convex Volatility Curves

- Recall that the yield to maturity definition arises from the benchmark bond pricing model with constant short interest rates, while the normal implied volatility definition arises from Bachelier's benchmark option pricing model with constant short (normal) variance rates.
- If the benchmark models are correct, then yields and implied volatilities are flat in term and moneyness respectively. In contrast, yields have historically been concave in term on average, while normal implied volatilities have historically been convex in moneyness on average.
- The names "yield frown" and "volatility smile" reflect the non-zero curvature of both graphs.
- For both the yield frown and the vol smile, we will present a pricing model which shows that their curvature arises from uncertainty in future yields and in future implied volatilities respectively.

Market Model for Yields

- We assume that the market gives us initial yields of zero coupon bonds at a finite number of maturities.
- The objective is to connect the dots, so as to produce a full yield curve.
- We assume no arbitrage and that $\mathbb{P}$ is the real world probability measure.
- Let r_t be the short interest rate whose dynamics are unspecified. Let $\mathbb{Q}$ be the martingale measure equivalent to $\mathbb{P}$, which arises when the money market account $e^{\int_0^t r_s ds}$ is taken to be the numeraire.
- Suppose that under $\mathbb{Q}$, the yield curve evolves continuously and only by parallel shifts:

$$dy_t(T) = \delta_t dt + \nu_t dZ_t, \qquad t \geq 0, \tag{1}$$

  where Z is a $\mathbb{Q}$ standard Brownian motion.
- Importantly, we do not need to specify the $\mathbb{Q}$ dynamics of the risk-neutral drift process δ_t or the yield volatility process ν_t when our only goal is to produce an entire arbitrage-free yield curve from a few given market quotes.

Market Model for Yields (Con'd)

- Let $b_t(T)$ be the market price of a bond. By the definition of yield to maturity $y_t(\tau)$:

$$b_t(T) = B^c(y_t(T), T - t), \qquad t \geq 0, T \geq t,$$

where recall the bond pricing function was defined as $B^c(y, \tau) = e^{-y\tau}, y \in \mathbb{R}, \tau \geq 0$.

- Itô's formula implies the following drift for $e^{-\int_0^t r_s ds} B^c(y_t(T), T - t)$:

$$E_t^{\mathbb{Q}} de^{-\int_0^t r_s ds} B^c(y_t(T), T-t) = \left[-r_t + \delta_t \frac{\partial}{\partial y} + \frac{\nu_t^2}{2} \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial t} \right] B^c(y_t(T), T-t)$$

- No arbitrage implies that this drift vanishes:

$$\left[-r_t + \delta_t \frac{\partial}{\partial y} + \frac{\nu_t^2}{2} \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial t} \right] B^c(y_t(T), T - t) = 0.$$

- The bond pricing formula B^c solves both this PDE and the special case when $0 = \delta_t = \nu_t = y_t(T) - r_t$. The introduction of two extra terms in the bond's drift is handled by letting y vary with both t and T.

Market Model for Yields (Con'd)

- Recall the no arbitrage constraint on yields implies that for $t \in [0, T]$:

$$\left[-r_t + \delta_t \frac{\partial}{\partial y} + \frac{\nu_t^2}{2}\frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial t}\right] B^c(y_t(T), T-t) = 0.$$

- From the bond pricing formula $B^c(y,\tau) = e^{-y\tau}, y \in \mathbb{R}, \tau \geq 0$, we have:
  1. $\frac{\partial}{\partial y} B^c(y_t(T), T-t) = -(T-t)B^c(y_t(T), T-t)$
  2. $\frac{\partial^2}{\partial y^2} B^c(y_t(T), T-t) = (T-t)^2 B^c(y_t(T), T-t)$
  3. $\frac{\partial}{\partial t} B^c(y_t(T), T-t) = y_t(T) B^c(y_t(T), T-t)$.
- Substituting these 3 greeks into the top eq'n & dividing out B^c implies:

$$y_t(T) = r_t + \delta_t(T-t) - \frac{\nu_t^2}{2}(T-t)^2, \qquad t \geq 0, T \geq t.$$

- Thus, when all yields move continuously and only by parallel shifts under $\mathbb{Q}$, the yield curve must be quadratic in term $T-t$, opening down.
- Notice that $y_t(T)$ is linear in r_t, δ_t, and ν_t^2. As a result, the market yields of 3 bonds uniquely determine the numerical values of the processes r_t, δ_t and ν_t^2. Once these values are known, the entire yield curve becomes known.

Market Model for Normal Implied Volatilities

- We now consider an entirely different model whose only objective is to price European options on some asset whose price is real-valued.
- We assume that the market gives us normal implied volatilities of co-terminal European options at a finite number of strikes. The objective is to connect the dots so as to produce a full (normal) implied volatility curve.
- We assume no arbitrage and zero interest rate. Let $S \in \mathbb{R}$ be the spot price of the option's underlying asset. Suppose that under $\mathbb{Q}$, S solves the following SDE:

$$dS_t = a_t dW_t, \qquad t \geq 0,$$

  where W is a $\mathbb{Q}$ standard Brownian motion.
- The stochastic process a is the instantaneous normal volatility of S. We do not directly specify a's dynamics.

Normal Implied Volatility

- Recall we are assuming that the underlying spot price S solves the SDE:

$$dS_t = a_t dW_t, \qquad t \geq 0,$$

where the normal volatility of S is the unspecified stochastic process a.

- Also recall that the concept of normal implied volatility arises from Bachelier's benchmark model which assumes in contrast that:

$$dS_t = a dW_t, \qquad t \geq 0, \text{ where } a \text{ is constant.}$$

- Let $\eta_t(K)$ be the normal IV by strike $K \in \mathbb{R}$ for fixed maturity date $T \geq t$.
- To compensate for not specifying the $\mathbb{Q}$ dynamics of a, we suppose that under $\mathbb{Q}$, the implied volatility curve moves continuously and that each IV experiences the same proportional shifts:

$$d\eta_t(K) = \omega_t \eta_t(K) dZ_t, \qquad K \in \mathbb{R}, t \geq 0,$$

where Z is a $\mathbb{Q}$ standard Brownian motion. The lognormal volatility ω_t of η_t is an unspecified but bounded stochastic process.

- We use proportional shifts for $\eta_t(K)$ so that all IV's stay positive.

Correlation and Covariation

- Recall the risk-neutral dynamics assumed for the underlying spot price S and the normal implied vol by strike $\eta_t(K)$:

$$dS_t = a_t dW_t, \qquad d\eta_t(K) = \omega_t \eta_t(K) dZ_t, \qquad t \geq 0,$$

  where W and Z are both univariate $\mathbb{Q}$ standard Brownian motions.

- Let $\rho_t \in [-1, 1]$ be the bounded stochastic process governing the correlation between increments of the two standard Brownian motions W and Z at time t: $d\langle W, Z\rangle_t = \rho_t dt$. Like a and ω, the stochastic process ρ is unspecified.
- The covariation between S and $\ln \eta_t(K)$ solves $d\langle S, \ln \eta(K)\rangle_t = \gamma_t dt$, where the covariation rate $\gamma_t \equiv a_t \rho_t \omega_t$ is independent of K.

No Arbitrage Condition for Normal Implied Vol

- Recall again the risk-neutral dynamics assumed for the underlying spot price S and the normal implied vol by strike $\eta_t(K), K \in \mathbb{R}$:

$$dS_t = a_t dW_t, \qquad d\eta_t(K) = \omega_t \eta_t(K) dZ_t, \qquad d\langle W, Z\rangle_t = \rho_t dt, \qquad t \geq 0.$$

- Recall that the Bachelier call value function C^b depends on spot S_t & strike K only through the excess $X_t = S_t - K$, which follows: $dX_t = a_t dW_t, t \geq 0$.
- By the definition of implied volatility, $c_t(K) = C^b(X_t, \eta_t(K), T - t)$, where $c_t(K)$ is the market price of the call at time $t \in [0, T]$ and:

$$C^b(x, \eta, \tau) \equiv \eta\sqrt{\tau} N'\left(\frac{x}{\eta\sqrt{\tau}}\right) + xN\left(\frac{x}{\eta\sqrt{\tau}}\right), \qquad x \in \mathbb{R}, \eta > 0, \tau > 0.$$

- No arbitrage implies that each call price $c_t(K)$ is a $\mathbb{Q}$ local martingale. From Itô's formula, implied volatilities $\eta_t(K), K \in \mathbb{R}$ solve:

$$\left[\frac{a_t^2}{2}\frac{\partial^2}{\partial x^2} + \gamma_t \eta_t(K)\frac{\partial^2}{\partial \eta \partial x} + \frac{\omega_t^2}{2}\eta_t^2(K)\frac{\partial^2}{\partial \eta^2} + \frac{\partial}{\partial t}\right] C^b(X_t, \eta_t(K), T - t) = 0,$$

where $\gamma_t \equiv \rho_t a_t \omega_t$ is the covariation rate between S and $\ln \eta_t(K)$.

No Arbitrage Condition for Normal Implied Vol (Con'd)

- Recall the implicit no arb. constraint on the IV curve $\eta_t(K), K \in \mathbb{R}$:

$$\left[\frac{a_t^2}{2}\frac{\partial^2}{\partial x^2} + \gamma_t \eta_t(K)\frac{\partial^2}{\partial \eta \partial x} + \frac{\omega_t^2}{2}\eta_t^2(K)\frac{\partial^2}{\partial \eta^2} + \frac{\partial}{\partial t}\right] C^b(X_t, \eta_t(K), T-t) = 0.$$

- The Bachelier call value function C^b solves both this PDE and the one with $0 = \gamma_t = \omega_t = \eta_t(K) - a_t$.
- Just as in the bond case, the introduction of two extra terms in the overlying's drift is handled by letting η vary with S and K.

No Arbitrage Condition for Normal Implied Vol (Con'd)

- Recall the implicit no arb. constraint on the IV curve $\eta_t(K), K \in \mathbb{R}$:

$$\left[\frac{a_t^2}{2}\frac{\partial^2}{\partial x^2} + \gamma_t \eta_t(K)\frac{\partial^2}{\partial \eta \partial x} + \frac{\omega_t^2}{2}\eta_t^2(K)\frac{\partial^2}{\partial \eta^2} + \frac{\partial}{\partial t}\right] C^b(X_t, \eta_t(K), T-t) = 0.$$

- Recall $\eta_t^2(K)/2$ can be seen as the rate of time decay in units of gamma:

$$\frac{\partial}{\partial t} C^b(X_t, \eta_t(K), T-t) = \frac{-\eta_t^2(K)}{2}\Gamma(X_t, \eta_t(K), T-t), \;\; K \in \mathbb{R}, t \in [0, T].$$

- The appendix proves that $\eta^n D_\eta^n D_x^{-n}\Gamma(x, \eta, \tau) = (-x)^n \Gamma(x, \eta, \tau), n = 0, 1, \ldots$

$$\begin{aligned} \text{For } n = 1: \eta \frac{\partial^2}{\partial \eta \partial x} C^b(x, \eta, \tau) &= \quad \eta D_\eta D_x^{-1}\Gamma(x, \eta, \tau) = \quad -x\Gamma(x, \eta, \tau) \\ \text{For } n = 2: \eta^2 \frac{\partial^2}{\partial \eta^2} C^b(x, \eta, \tau) &= \quad \eta^2 D_\eta^2 D_x^{-2}\Gamma(x, \eta, \tau) = \quad x^2\Gamma(x, \eta, \tau). \end{aligned}$$

- Substituting the 3 greek rel'ns in the top eqn. and dividing out Γ implies:

$$\frac{\eta_t^2(K)}{2} = \frac{a_t^2}{2} + \gamma_t(K - S_t) + \frac{\omega_t^2}{2}(K - S_t)^2, \qquad K \in \mathbb{R}.$$

Quadratic Normal Implied Variance Rate Curve

- Recall the no arbitrage condition for the normal IV curve, $\eta_t(K), K \in \mathbb{R}$:

$$\frac{\eta_t^2(K)}{2} = \frac{a_t^2}{2} + \gamma_t(K - S_t) + \frac{\omega_t^2}{2}(K - S_t)^2, \qquad K \in \mathbb{R}.$$

- When S evolves arithmetically while all normal implied volatilities $\eta(K), K \in \mathbb{R}$ experience the same proportional shocks, the halved implied variance rate curve is quadratic in moneyness $K - S$, opening up.
- It is straightforward to use the quadratic root formula on the top equation to determine how normal IV, $\eta_t(K)$, depends on the moneyness, $K - S$.
- Notice that $\frac{\eta_t^2(K)}{2}$ is linear in a_t^2, γ_t, and ω_t^2. As a result, the market quotes of 3 co-terminal normal implied volatilities uniquely determine the numerical values of a_t^2, γ_t, and ω_t^2, and hence a_t, ρ_t and ω_t.
- Once these values are known, the entire halved implied variance curve becomes known, despite zero knowledge of how the 3 processes will evolve.

Implied Variance is a Variance!

- Recall the no arbitrage condition for the normal IV curve, $\eta_t(K), K \in \mathbb{R}$:

$$\eta_t^2(K) = a_t^2 + 2\gamma_t(K - S_t) + \omega_t^2(K - S_t)^2, \qquad K \in \mathbb{R}.$$

- Now $a_t^2 dt = (dS_t)^2$, $\gamma_t dt = dS_t d \ln \eta_t(K)$, and $\omega_t^2 dt = (d \ln \eta_t(K))^2, K \in \mathbb{R}$.
- As a result, the implied variance rate at strike $K \in \mathbb{R}$ IS a variance:

$$\eta_t^2(K)dt = \mathrm{Var}_t^{\mathbb{Q}}(dS_t + (K - S)d \ln \eta_t(K))\Big|_{S=S_t}.$$

- Implied variance is actually the right variance to put into the right model to reach the right price.

Comparing Market Models

- The arbitrage-free yield frown that arises when all yields are driven by a single standard Brownian motion (SBM) and move only by parallel shifts:

$$y_t(T) = r_t + \delta_t(T-t) - \nu_t^2 \frac{(T-t)^2}{2}, \qquad T \geq t \geq 0,$$

can be compared to the arbitrage-free halved implied variance smile that arises when spot and implied volatilities are driven by correlated SBM's and all implied volatilities experience the same proportional shifts:

$$\frac{\eta_t^2(K)}{2} = \frac{a_t^2}{2} + \gamma_t(K - S_t) + \omega_t^2 \frac{(K-S_t)^2}{2}, \qquad K, S_t \in \mathbb{R}.$$

- Both curves have 3 components. For yields, the intercept is the short rate, r_t the slope in term is the yield drift δ_t, while the curvature in term is $-\nu_t^2$.
- For halved implied variance rates, the intercept is the halved short variance rate, $\frac{a_t^2}{2}$, the slope in moneyness is the covariation rate γ_t, while the positive curvature in moneyness is the lognormal variance rate of IV, ω_t^2.
- The different signs for curvature arise because yields are decreasing in bond prices, while halved implied variances are increasing in option prices.

Relative Robustness and Limited Scope of Market Models

- Recall again our arbitrage-free yield frown:
$$y_t(T) = r_t + \delta_t(T-t) - \nu_t^2 \frac{(T-t)^2}{2}, \qquad T \geq t \geq 0,$$
and our arbitrage-free halved implied variance smile:
$$\frac{\eta_t^2(K)}{2} = \frac{a_t^2}{2} + \gamma_t(K - S_t) + \omega_t^2 \frac{(K-S_t)^2}{2}, \qquad K, S_t \in \mathbb{R}.$$
- Note that the random variation over time of the coefficients in term $T-t$ and moneyness $K-S$ is entirely consistent with the market model. This consistency is in stark contrast to parameter variation in short rate models. Systematic parameter variation over time in short rate models requires an alternative dynamical specification, which will in general lead to a different functional form for the yield or IV curve.
- While market models enjoy this advantage for the problem of curve construction, they can only be used to value bonds or options (and linear combinations thereof such as coupon bonds and path-independent payoffs). In contrast, a more standard stochastic short rate model can be used to value path-dependent derivatives consistently.

Summary and Conclusions

- Practitioners and academics have both recognized that variance rates play a similar role in option pricing as interest rates do in bond pricing.
- The term structure of interest rates indicates the theta of each 1$ investment in bonds. Analogously, the moneyness structure of halved implied variance rates indicates the negated theta of each unit gamma position in options.
- In this presentation, we imposed particular risk-neutral dynamics for the yield curve and the normal implied vol curve, so that the resulting arb-free yield frown is analogous to the resulting arb-free halved implied variance smile.
- Market models were used to develop quadratic arbitrage-free curves in both cases.
- Thanks for listening (despite the variance in interest).

In this appendix, we provide a short proof that for any sufficiently differentiable function $f : \mathbb{R} \mapsto \mathbb{R}$ and for $n = 0, 1, \ldots$:

$$(D_s D_x^{-1})^n \frac{f\left(\frac{x}{s}\right)}{s} = \left(-\frac{x}{s}\right)^n \frac{f\left(\frac{x}{s}\right)}{s}, \qquad s > 0, x \in \mathbb{R}. \tag{2}$$

We first show the result holds for $n = 1$, i.e.
$D_s D_x^{-1} \frac{f\left(\frac{x}{s}\right)}{s} = \left(-\frac{x}{s}\right) \frac{f\left(\frac{x}{s}\right)}{s}, s > 0, x \in \mathbb{R}$. The LHS is:
$D_s D_x^{-1} \frac{f\left(\frac{x}{s}\right)}{s} = D_s \int\limits_{-\infty}^{x} \frac{f\left(\frac{y}{s}\right)}{s} dy = D_s \int\limits_{-\infty}^{\frac{x}{s}} f(z)dz = \frac{-\frac{x}{s} f\left(\frac{x}{s}\right)}{s}$, by the fundamental theorem of calculus and the chain rule. Thus, for $n = 1$, the result does hold for any fraction $\frac{f(z)}{s}, z = \frac{x}{s}$. Notice that the effect of applying the operator $D_s D_x^{-1}$ to the fraction $\frac{f(z)}{s}, z = \frac{x}{s}$, is another fraction $\frac{g(z)}{s}, z = \frac{x}{s}$, where $g(z) \equiv -zf(z)$. As a result, one can apply the operator $D_s D_x^{-1}$ to the fraction $\frac{g}{s}$ to obtain:

$$\left(D_s D_x^{-1}\right)^2 \frac{f\left(\frac{x}{s}\right)}{s} = \frac{-\frac{x}{s} g\left(\frac{x}{s}\right)}{s} = \frac{\left(-\frac{x}{s}\right)^2 f\left(\frac{x}{s}\right)}{s}. \tag{3}$$

Repeating this exercise $n - 2$ times leads to the desired result (2). Re-arranging (2) implies that for any sufficiently differentiable function $f : \mathbb{R} \mapsto \mathbb{R}$:

$$s^n D_s^n D_x^{-n} \frac{f\left(\frac{x}{s}\right)}{s} = (-x)^n \frac{f\left(\frac{x}{s}\right)}{s}, \qquad s > 0, x \in \mathbb{R}, n = 0, 1, \ldots. \qquad \textbf{Q.E.D.} \tag{4}$$